4.8 phase plane analysis

Wednesday, March 10, 2021 3:41 AM

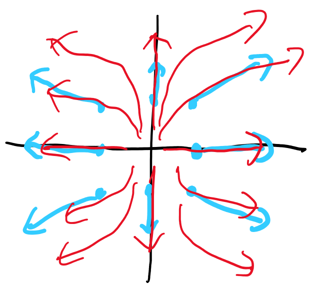
Consider an autonomous, homogeneous 2D linear system with constant coefficients

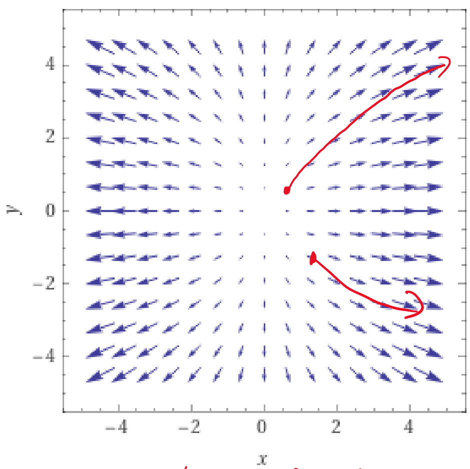
$$\dot{X}(t) = A X(t), \qquad A \in \mathbb{R}^{2\times 2}$$

The system defines a vector field, associating a direction and magnitude for every point in $\mathbb{R}^2$

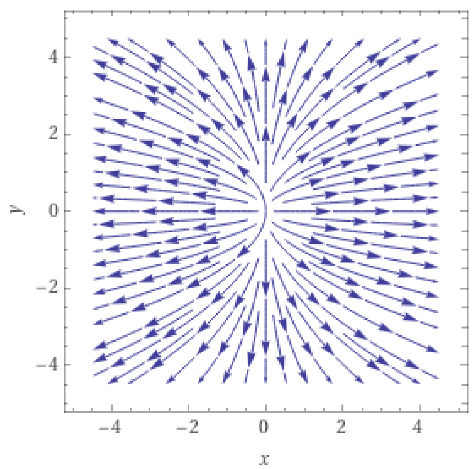
Ex. $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$X(t) = c_1 e^{2t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{t}\begin{bmatrix}0\\1\end{bmatrix}$$





vector field



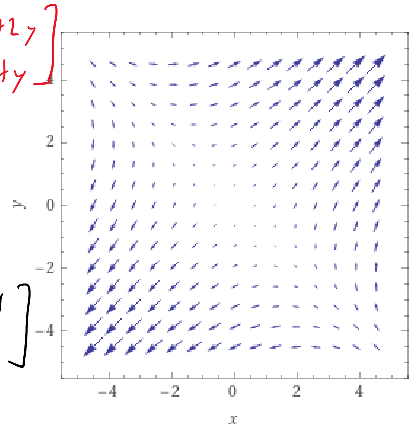
integral curves

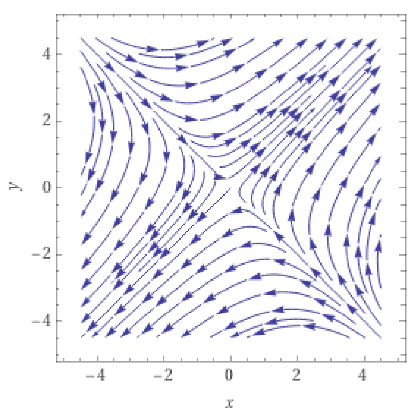
Ex. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ 2x+y \end{bmatrix}$

$\lambda_1 = -1 \qquad V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3 \qquad V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

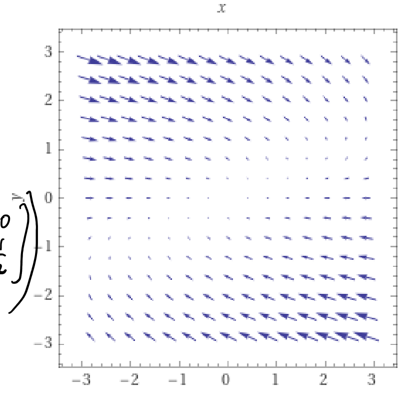
$$X(t) = c_1 e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\1\end{bmatrix}$$

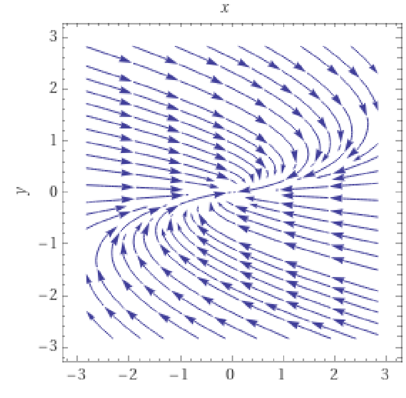




$A = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$

$$X(t) = c_1 e^{-t}\begin{bmatrix}1\\0\end{bmatrix} + c_2\left(te^{-t}\begin{bmatrix}1\\0\end{bmatrix} + e^{-t}\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}\right)$$





Equilibrium solutions

A point $\bar{X} \in \mathbb{R}^2$ is an equilibrium solution if the constant function

$$\bar{X}(t) = \bar{X}$$ is a solution to the system.

Thus $\dfrac{d\bar{X}}{dt} = A\bar{X} \implies A\bar{X} = 0 \implies \bar{X} = 0$ is an equilibrium.

Also, if A is invertible, then $\bar{X} = 0$ is the unique equilibrium.

If all eigenvalues of A have negative real part, then $\lim_{t\to\infty} X(t) = 0$ for all solutions $X(t)$, so the origin is **asymptotically stable**.

The origin is **stable** if all eigenvalues have nonpositive real part.

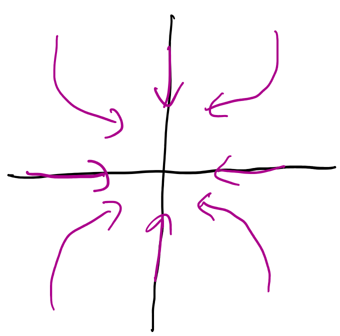
The origin is **unstable** if at least one eigenvalue has positive real part.

Real Eigenvalues: Nodes

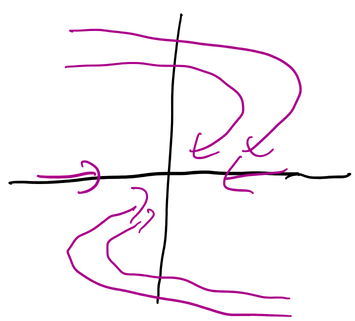
Def. The origin of $\dot{X} = AX$, $A \in \mathbb{R}^{2\times 2}$, is a **node** if both eigenvalues λ_1, λ_2 have the same sign and are real.

Def. A node is **proper** if $\lambda_1 = \lambda_2$ and there are two linearly independent eigenvectors. It is **improper** otherwise.

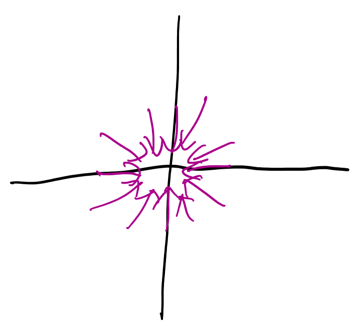
Ex.



Stable improper node



Stable improper node



stable proper node

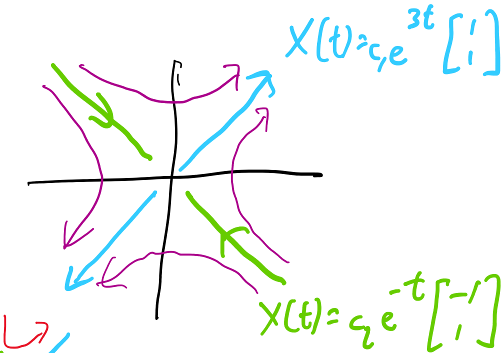
Real eigenvalues: Saddles

Def. The origin is a **saddle pt** if $\lambda_1, \lambda_2 \in \mathbb{R}$ and have opposite signs. WLOG, say $\lambda_1 < 0 < \lambda_2$.

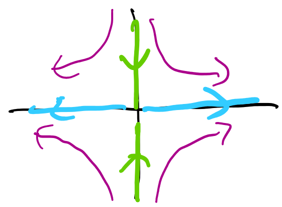
Note: Since $\lambda_1 \neq \lambda_2$, we have two linearly ind. eigenvectors V_1, V_2.

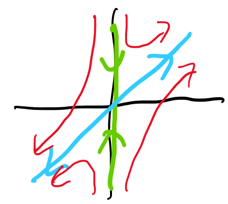
Then $X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \qquad \begin{array}{ll} \lambda_1 = -1 & V_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \lambda_2 = 3 & V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array}$$



$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$





Complex eigenvalues

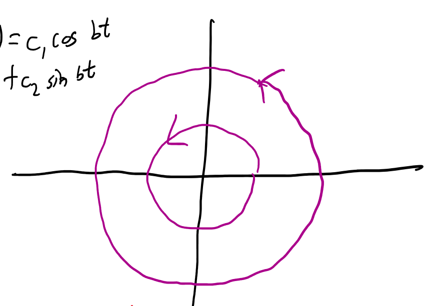
Def. If $\lambda_1 = a + bi$, $\lambda_2 = a - bi$, $a, b \in \mathbb{R}$, then the origin is a **spiral (or focus)** if $a \neq 0$, and a **center** if $a = 0$.

If $a < 0$, then it is an **asymptotically stable** spiral.

If $a > 0$, then it is an **unstable** spiral.

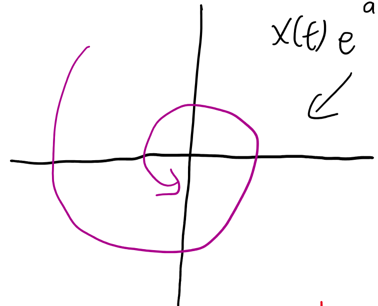
A center is **(neutrally) stable** (but not asymptotically stable)

$X(t) = c_1 \cos bt + c_2 \sin bt$

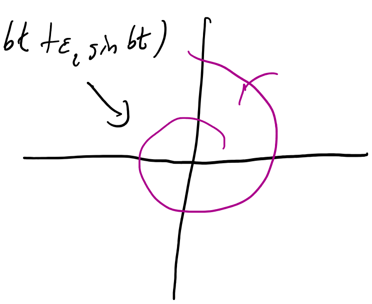


center

$X(t)\, e^{at}(c_1 \cos bt + c_2 \sin bt)$



(asympt.) stable spiral



unstable spiral

Stability diagram

Let $A \in \mathbb{R}^{2\times 2}$, $\tau = Tr(A)$, $\delta = det(A)$

Then $\lambda^2 - \tau\lambda + \delta$ is the characteristic polynomial of A.

$\Rightarrow \lambda_{1,2} = \dfrac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$, where $\gamma = \tau^2 - 4\delta$ is the discriminant.

If $\gamma \geq 0$, eigenvalues are real.

- If $\tau > 0$ and $\delta > 0$, then $\lambda_1 > 0$, $\lambda_2 > 0$, so unstable node
- If $\delta < 0$, then $\lambda_1 \lambda_2 < 0$, so saddle pt.
- If $\tau < 0$ and $\delta < 0$, then $\lambda_1 < 0$, $\lambda_2 < 0$, so stable node.

If $\gamma < 0$, eigenvalues are complex

- If $\tau > 0$, then $Re(\lambda_{1,2}) > 0$, so unstable spiral.
- If $\tau = 0$, then $Re(\lambda_{1,2}) = 0$, so neutral center.
- If $\tau < 0$, then $Re(\lambda_{1,2}) < 0$, so stable spiral.

Stability diagram